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1995 J. Phys. A: Math. Gen. 28 4305

(http://iopscience.iop.org/0305-4470/28/15/011)

J. Phys. A: Math. Gen. 28 (1995) 4305–4311. Printed in the UK

Path integral representation for fractional Brownian motion

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Received 15 March 1995

Abstract. Fractional Brownian motion (FBM) is a generalization of the usual Brownian motion. A path integral representation that has recently been suggested for it is shown to be not for the FBM but for a different generalization of the Brownian motion. A new path integral representation is given and its measure has fractional derivatives of the path in it. The measure shows that the process is Gaussian but is, in general, non-Markovian, even though Brownian motion itself is Markovian. It is shown how the propagator for the motion of free FBM may be evaluated. This is somewhat more complex than for the usual path integrals, due to the occurrence of fractional derivatives. We also find the propagator in the presence of a linear absorption (potential), and for FBM on a ring.

1. Introduction

Brownian motion has served as a model for several problems in chemical physics. The path (Wiener) integral representation of the time evolution of its probability distribution function is quite well known. A generalization of the Brownian motion, referred to as fractional Brownian motion (FBM) has also been suggested and is being recognized as a model of wide utility (Mandelbrot and van Ness 1968, Fan *et al* 1991, Cherayil and Biswas 1993, Giona and Roman 1992, Roman and Giona 1992). However, the correct path integral representation of the probability distribution function for the FBM does not seem to exist in the literature.

In one dimension, the motion is defined as follows. The position $x(t)$ of the walker at the time t, who started at X_0 at the time $t = 0$, is given by

$$\begin{aligned} x(t) - X_0 &= \mathrm{D}^{-\alpha}\xi(t) \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-t_1)^{\alpha-1}\xi(t_1)\,\mathrm{d}t_1. \end{aligned} \tag{1}$$

α is a parameter that may take non-integral values greater than zero. $\mathrm{D} = \mathrm{d}/\mathrm{d}t$ and $\mathrm{D}^{-\alpha}$ is the fractional integral of order α (Widder 1971, Srivastava and Manocha 1984) defined by the second line in equation (1). $\xi(t)$ is the white noise, having the autocorrelation function $\langle\xi(t)\xi(s)\rangle = \delta(t-s)$. The usual Brownian motion would correspond to $\alpha = 1$. *Thus* $x(t) - X_0$ *is the fractional integral of order* α, *of the usual white noise.* Cherayil and Biswas (1993) have suggested that the path followed by the three-dimensional version of the FBM may be taken as a model for the configuration of a polymer molecule in solution. With this aim, a path integral representation for FBM has been proposed by them. Their analysis is based upon the results of Maccone (1981a, b) who gave the following expression for the autocorrelation function (in our notation):

$$C_{\mathrm{M}}^{\alpha}(t,s)=\frac{1}{(2\alpha-1)\Gamma^2(\alpha)}[\min(s,t)]^{2\alpha-1}. \tag{2}$$

The subscript M stands for Maccone. Note that our α is equal to $(h+1/2)$ of Maccone (1981a, b). Cherayil and Biswas (1993) continued this work and developed a path integral representation for a Brownian motion which has this autocorrelation function. While their development is fully correct, $C_{\mathrm{M}}^{\alpha}(t,s)$ is not the correct autocorrelation function for the FBM. Using the definition of equation (1), we find the correct autocorrelation function as below:

$$\begin{aligned} C^{\alpha}(t,s) &= \langle x(t)x(s)\rangle \\ &= \frac{1}{\Gamma^2(\alpha)}\int_0^t \mathrm{d}t_1 \int_0^s \mathrm{d}s_1\,(t-t_1)^{\alpha-1}(s-s_1)^{\alpha-1}\delta(t_1-s_1). \end{aligned} \tag{3}$$

Defining $t_< = \min(t,s)$ and $t_> = \max(t,s)$, we can write

$$C^{\alpha}(t,s)=\frac{1}{\Gamma^2(\alpha)}\int_0^{t_<}\mathrm{d}t_1\,(t-t_1)^{\alpha-1}(s-t_1)^{\alpha-1} \tag{4}$$

$$=\frac{t_>^{\alpha-1}t_<^{\alpha}}{\alpha\Gamma^2(\alpha)}\,{}_2F_1(1,1-\alpha;1+\alpha;t_</t_>). \tag{5}$$

${}_2F_1$ is the hypergeometric function (Magnus *et al* 1966). If $t=s$ and $\alpha>1/2$, then using the value ${}_2F_1(1,1-\alpha;1+\alpha;1)=\alpha/(2\alpha-1)$ gives

$$C^{\alpha}(t,t)=\frac{1}{(2\alpha-1)\Gamma^2(\alpha)}t^{2\alpha-1}. \tag{6}$$

If $\alpha\leqslant 1/2$, then the equal time correlation function diverges, as may be seen directly from equation (4). Therefore, we shall consider only the case where $\alpha>1/2$. Furthermore, we shall put the restriction $\alpha<1$ in this paper. The case $\alpha>1$ is discussed briefly towards the end of section 2. As the correct correlation function of equation (4) is more complicated than the one given by Maccone (1981a), the path integral representation for the FBM is more complex than the one given by Cherayil and Biswas (1993). However, a generalization of Brownian motion $x_{\mathrm{M}}(t)$ defined by

$$x_{\mathrm{M}}(t)-X_0=\frac{1}{\Gamma(\alpha)}\int_0^t t_1^{\alpha-1}\xi(t_1)\,\mathrm{d}t_1 \tag{7}$$

has the autocorrelation function of Maccone ($C_{\mathrm{M}}^{\alpha}(t,s)$). Therefore, the path integral representation and the associated diffusion equation developed by Cherayil and Biswas (1993) are for $x_{\mathrm{M}}(t)$ and not for the one defined by equation (1).

As the FBM is of considerable interest, we now proceed to construct the path integral representation for it. For this, we formally invert the equation (1) to get

$$\mathrm{D}^{\alpha}\delta x(t)=\xi(t). \tag{8}$$

We have adopted the notation $\delta x(t)=x(t)-X_0$. D^{α} is the fractional derivative (Widder 1971, Srivastava and Manocha 1984) and is defined by $\mathrm{D}^{\alpha}=\mathrm{D}\,\mathrm{D}^{\alpha-1}$ for $0<\alpha<1$. Note also that this is not the same as $\mathrm{D}^{\alpha-1}\mathrm{D}$, as may be seen by allowing the two operators to operate upon any function that is a constant. As the measure in the path integral representation for the white noise $\xi(t)$ has $\exp\{-\frac{1}{2}\int \mathrm{d}t\, \xi^2(t)\}$ in it, and as $x(t)$ is linearly related to $\xi(t)$, the measure for $x(t)$ is clearly proportional to $\exp\{-\frac{1}{2}\int \mathrm{d}t\, [\mathrm{D}^\alpha \delta x(t)]^2\}$. So we write the probability of finding the particle at X_T at the time T as

$$G(X_T, T|X_0, 0) = \int_{0,X_0}^{T,X_T} \mathrm{D}x(t) \exp\left\{ -\frac{1}{2}\int_0^T \mathrm{d}t\, [\mathrm{D}^\alpha \delta x(t)]^2 \right\} \tag{9}$$

which is the desired path integral representation for the FBM. We now show how this propagator and that for certain other cases may be calculated.

2. Propagator for the free particle

The path integral of equation (9) may be evaluated by techniques similar to the ones used for the usual path integrals (Feynman and Hibbs 1965). The procedure is somewhat more complex than usual, due to the occurrence of the fractional derivatives in the expressions, and therefore we give the details. As the first step in the evaluation, we change over to the new path variable $y(t)$, defined by

$$\delta x(t) = \delta \bar{x}(t) + y(t) \tag{10}$$

where $\delta \bar{x}(t)$ is the 'extremum path', for which the 'action'

$$S = \frac{1}{2}\int_0^T \mathrm{d}t\, [\mathrm{D}^\alpha \delta x(t)]^2$$

is an extremum. It is taken to satisfy the boundary conditions

$$\delta \bar{x}(0) = 0 \qquad \text{and} \qquad \delta \bar{x}(T) = X_T - X_0. \tag{11}$$

Therefore, the path variable $y(t)$ has to obey $y(0) = y(T) = 0$. Substituting equation (10) into the action, one finds the extremum path by putting the term linear in $y(t)$ to be equal to zero. The term is found to be

$$\delta S = \int_0^T \mathrm{d}t\, \mathrm{D}^\alpha \delta \bar{x}(t)\, \mathrm{D}^\alpha y(t). \tag{12}$$

Using the definition $\mathrm{D}^\alpha = \mathrm{D}\,\mathrm{D}^{\alpha-1}$, and integrating by parts, noting that $\mathrm{D}^{\alpha-1}y(0) = 0$, leads to

$$\delta S = -\int_0^T \mathrm{d}t\, \mathrm{D}^{\alpha-1}y(t)\, \mathrm{D}^{1+\alpha}\delta \bar{x}(t) + \mathrm{D}^\alpha \delta \bar{x}(T)\, \mathrm{D}^{\alpha-1}y(T). \tag{13}$$

Remembering the definition of

$$\mathrm{D}^{\alpha-1}y(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \mathrm{d}t_1\, (t-t_1)^{-\alpha} y(t_1)$$

we can rearrange the right-hand side to give

$$\delta S = -\int_0^T \mathrm{d}t\, y(t) \int_t^T \mathrm{d}s \frac{(s-t)^{-\alpha}}{\Gamma(1-\alpha)} \mathrm{D}^{1+\alpha}\delta\bar{x}(s) + \mathrm{D}^{\alpha}\delta\bar{x}(T) \int_0^T \mathrm{d}t\, \frac{(T-t)^{-\alpha}}{\Gamma(1-\alpha)} y(t). \tag{14}$$

If $\delta\bar{x}(t)$ is to be the extremum path, then δS should vanish for arbitrary $y(t)$. Hence

$${}_T\mathrm{D}^{\alpha-1}\mathrm{D}^{1+\alpha}\delta\bar{x}(t) = \mathrm{D}^{\alpha}\delta\bar{x}(T)\frac{(T-t)^{-\alpha}}{\Gamma(1-\alpha)} \tag{15}$$

where ${}_T\mathrm{D}^{\alpha-1}$ is defined by

$${}_T\mathrm{D}^{\alpha-1} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_t^T \mathrm{d}t_1\, (t_1 - t)^{-\alpha} f(t_1) \tag{16}$$

for any $f(t)$. The change in the path variable in the equation (10) leads to

$$\begin{aligned} G(X_T, T|X_0, 0) &= \exp\left\{ -\tfrac{1}{2} \int_0^T \mathrm{d}t\, [\mathrm{D}^{\alpha}\delta\bar{x}(t)]^2 \right\} \int_{0,0}^{0,T} \mathrm{D}y(t) \exp\left\{ -\tfrac{1}{2}\int_0^T \mathrm{d}t\, [\mathrm{D}^{\alpha} y(t)]^2 \right\} \\ &= \exp\left\{ -\tfrac{1}{2} \int_0^T \mathrm{d}t\, [\mathrm{D}^{\alpha}\delta\bar{x}(t)]^2 \right\} G(0, T|0, 0). \end{aligned} \tag{17}$$

2.1. The extremum path $\delta\bar{x}(t)$

We now solve equation (15), subject to the boundary conditions of (11). Using definition (16), one can easily prove that ${}_T\mathrm{D}^{-\alpha}\,{}_T\mathrm{D}^{\alpha-1} = {}_T\mathrm{D}^{-1}$. Therefore, we operate on both sides of equation (15) with ${}_T\mathrm{D}^{-\alpha}$. As this is an integral operator, it is only natural that one has to add something similar to the constant that results in the usual integration D^{-1}. The term that is to be added here, however, is not a constant, but the function $-C(T-t)^{\alpha-1}$, where C is a constant. It may easily be verified that ${}_T\mathrm{D}^{\alpha}(T-t)^{\alpha-1} = 0$. Therefore,

$${}_T\mathrm{D}^{-1}\,\mathrm{D}^{1+\alpha}\delta\bar{x}(t) = \mathrm{D}^{\alpha}\delta\bar{x}(T)\frac{{}_T\mathrm{D}^{-\alpha}(T-t)^{-\alpha}}{\Gamma(1-\alpha)} - C(T-t)^{\alpha-1}. \tag{18}$$

On evaluation, ${}_T\mathrm{D}^{-\alpha}(T-t)^{-\alpha} = \Gamma(1-\alpha)$. Hence equation (18) may be written as

$$\int_t^T \mathrm{d}t\, \mathrm{D}^{1+\alpha}\delta\bar{x}(t) = \mathrm{D}^{\alpha}\delta\bar{x}(T) - C(T-t)^{\alpha-1} \tag{19}$$

resulting in

$$\mathrm{D}^{\alpha}\delta\bar{x}(t) = C(T-t)^{\alpha-1}. \tag{20}$$

We now apply $\mathrm{D}^{-\alpha}$ to both sides to get

$$\delta\bar{x}(t) = C\,\mathrm{D}^{-\alpha}(T-t)^{\alpha-1} + C_1 t^{\alpha-1}. \tag{21}$$

C_1 is a constant. Written in detail,

$$\delta\bar{x}(t) = \frac{C}{\Gamma(\alpha)} \int_0^t \mathrm{d}s\, (t-s)^{\alpha-1}(T-s)^{\alpha-1} + C_1 t^{\alpha-1}. \tag{22}$$

The use of the boundary conditions of equation (11) gives

$$C_1 = 0 \qquad \text{and} \qquad C = (X_T - X_0)\Gamma(\alpha)(2\alpha - 1)T^{1-2\alpha}. \tag{23}$$

Hence,

$$\delta\bar{x}(t) = (X_T - X_0)(2\alpha - 1)T^{1-2\alpha}\int_0^t \mathrm{d}s\,(t-s)^{\alpha-1}(T-s)^{\alpha-1} \tag{24}$$

is the classical path. Equations (23) and (20) may be used to get the action for the classical path to be

$$\frac{1}{2}\int_0^T \mathrm{d}t\,[\mathrm{D}^\alpha\delta\bar{x}(t)]^2 = \frac{(X_T - X_0)^2\Gamma^2(\alpha)(2\alpha-1)}{2T^{2\alpha-1}}. \tag{25}$$

Using this result in (9), we get the propagator

$$G(X_T, T|X_0, 0) = \exp\left\{-\frac{(X_T - X_0)^2\Gamma^2(\alpha)(2\alpha-1)}{2T^{2\alpha-1}}\right\}G(0, T|0, 0). \tag{26}$$

Now it is an easy matter to find $G(0, T|0, 0)$—it has to be chosen so that $\int_{-\infty}^{\infty}\mathrm{d}X_T\,G(X_T, T|X_0, 0) = 1$. This gives

$$G(X_T, T|X_0, 0) = \left(\frac{(2\alpha-1)\Gamma^2(\alpha)}{2T^{2\alpha-1}\pi}\right)^{1/2}\exp\left\{-\frac{(X_T - X_0)^2\Gamma^2(\alpha)(2\alpha-1)}{2T^{2\alpha-1}}\right\}. \tag{27}$$

This result, however, is identical to that of Cherayil and Biswas (1993). This is not at all surprising—the propagator for a Gaussian random function is determined only by the equal time correlation function, which is the same for $x(t)$ and $x_{\mathrm{M}}(t)$. However, the detailed behaviour of our paths and those of Cherayil and Biswas (1993) will be very different, as may be shown by calculating the high-order correlation functions (Sebastian 1994). Also, using the fact that $\delta x(0) = 0$, one can show that $\mathrm{D}^\alpha\delta x(t) = \mathrm{D}^{\alpha-1}\mathrm{D}\,\delta x = \mathrm{D}^{\alpha-1}\mathrm{d}x/\mathrm{d}t$. Hence, adopting the notation $\dot{x} = \mathrm{d}x/\mathrm{d}t$, one can rewrite the measure of the path integral as $\exp\{-\frac{1}{2}\int_0^T \mathrm{d}t\,[\mathrm{D}^{\alpha-1}\dot{x}(t)]^2\}$, *which makes its connection with the usual path integrals* clearer. Written in more detail it is $\exp\{-\frac{1}{2}\int_0^T \mathrm{d}s\int_0^T \mathrm{d}s_1\,\mathfrak{C}(s, s_1)\,\dot{x}(s)\,\dot{x}(s_1)\}$ where

$$\mathfrak{C}(s, s_1) = \frac{1}{\Gamma^2(1-\alpha)}\int_0^T \mathrm{d}t\,(t-s)^{-\alpha}(t-s_1)^{-\alpha}\theta(t-s)\theta(t-s_1).$$

This means that the function $x(t)$ is not Markovian unless $\alpha = 1$, which is to be compared with the fact that the function of Cherayil and Biswas (1993) is Markovian.

Now we consider the case $2 > \alpha > 1$ briefly. For this, the basic equation $\mathrm{D}^\alpha x(t) = \xi(t)$ may be written as $\mathrm{D}\,\mathrm{D}^{\alpha-1}x(t) = \xi(t)$ and on integration,

$$x(t) = x(0) + \mathrm{D}^{\alpha-1}x(0)/\Gamma(\alpha) + \int_0^t \mathrm{d}s\,\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\xi(s). \tag{28}$$

Therefore the walk is specified only by giving values of $x(0)$ and $\mathrm{D}^{\alpha-1}x(0)$, in contrast to the case where $\alpha \leqslant 1$, where only $x(0)$ is enough. We shall not discuss this further in this paper.

3. Propagator for FBM with linear absorption

In this section, we consider the path integral

$$G^l(X_T, T|X_0, 0) = \int_{0,X_0}^{T,X_T} \mathrm{D}x(t) \exp\left\{ -\tfrac{1}{2}\int_0^T \mathrm{d}t\,[\mathrm{D}^\alpha \delta x(t)]^2 - \int_0^T \mathrm{d}t\, p(t)x(t)\right\} \tag{29}$$

which corresponds to absorption of the particle at a rate proportional to x at the position x, the proportionality factor being time dependent. As before, finding the classical path leads to

$$\mathrm{D}^\alpha \delta\bar{x}(t) = {}_T\mathrm{D}^{-\alpha} p(t) + C(T-t)^{\alpha-1}. \tag{30}$$

Written in more detail, this is

$$\mathrm{D}^\alpha \delta\bar{x}(t) = \frac{1}{\Gamma(\alpha)}\int_t^T \mathrm{d}s\,(t-s)^{\alpha-1} p(s) + C(T-t)^{\alpha-1}. \tag{31}$$

This equation is somewhat different from the usual equations in that the value of $\mathrm{D}^\alpha\,\delta\bar{x}(t)$ at the time t depends on the behaviour of the absorbing term $p(s)$ for $t < s < T$ and hence the equation is not causal. This again is not surprising—this is a general feature of this type of path integral (Sebastian 1982). We now consider the case where $p(s) =$ a constant, which we denote as p. Then we get

$$\delta\bar{x}(t) = \frac{C}{\Gamma(\alpha)}\int_0^t \mathrm{d}s\,(t-s)^{\alpha-1}(T-s)^{\alpha-1} - \frac{p}{\Gamma(\alpha)\Gamma(1+\alpha)}\int_0^t \mathrm{d}s\,(t-s)^{\alpha-1}(T-s)^{\alpha} + C_1 t^{\alpha-1}. \tag{32}$$

$\delta\bar{x}(0) = 0$ implies $C_1 = 0$ and putting the condition $\delta\bar{x}(T) = X_T - X_0$ enables us to determine the value of C. Making the same change as in equation (10) gives

$$G^l(X_T, T|X_0, 0) = \exp\left\{ -\tfrac{1}{2}\int_0^T \mathrm{d}t\,[\mathrm{D}^\alpha \delta\bar{x}(t)]^2 - \int_0^T \mathrm{d}t\, p\bar{x}(t)\right\} G(0, T|0, 0). \tag{33}$$

Evaluating the action for the classical path then gives

$$G^l(X_T, T|X_0, 0) = G(X_T, T|X_0, 0)\exp\{-a_1 p - a_2 p^2\}. \tag{34}$$

with

$$a_1 = \{X_0(1-2\alpha+2\alpha^2) - X_T(1-2\alpha)\}\frac{T}{2\alpha^2} \qquad \text{and} \qquad a_2 = -\frac{T^{1+2\alpha}}{8\alpha^2(1+2\alpha)\Gamma^2(1+\alpha)}.$$

4. FBM on a ring

With the expression for the propagator for FBM on a line, it is an easy matter to write the propagator for a ring. It is

$$G(\theta_T, T|\theta_0, 0) = \left(\frac{(2\alpha-1)\Gamma^2(\alpha)}{2T^{2\alpha-1}\pi}\right)^{1/2} \sum_{n=-\infty}^{\infty} \exp\left\{-\frac{(\theta_T - \theta_0 + 2n\pi)^2\Gamma^2(\alpha)(2\alpha-1)}{2T^{2\alpha-1}}\right\} \tag{35}$$

θ is the angle coordinate and n is the winding number of the path around the ring.

After the completion of this work, it was pointed out to me that the correct correlation function is also evaluated in a paper by Wyss (1991).

Acknowledgments

I wish to thank Professor S K Rangarajan, to whom I owe so much. I am grateful to the Alexander von Humboldt Foundation for supporting my work by the gift of a computer system.

References

Cherayil B J and Biswas P 1993 *J. Chem. Phys.* **99** 9230
Fan L T, Neogi D and Yashima M 1991 *Elementary Introduction to Spatial and Temporal Fractals (Lecture Notes in Chemistry 55)* ed G Berthier (Berlin: Springer)
Feynman R P and Hibbs A R 1965 *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill)
Giona M and Roman H E 1992 *J. Phys. A: Math. Gen.* **25** 2093
Maccone G 1981a *Nuovo Cimento* B **61** 229
—— 1981b *Nuovo Cimento* B **65** 259
Magnus M, Oberhettinger F and Soni R P 1966 *Formulas and Theorems for the Special Functions of Mathematical Physics* (New York: Springer)
Mandelbrot B B and van Ness J W 1968 *SIAM Rev.* **10** 422
Roman H E and Giona M 1992 *J. Phys. A: Math. Gen.* **25** 2107
Sebastian K L 1982 *Phys. Lett.* **91A** 387
—— 1994 unpublished results
Srivastava H M and Manocha H L 1984 *A Treatise on Generating Functions* (Chichester: Ellis Horwood)
Widder D V 1971 *An Introduction to Transform Theory* (New York: Academic)
Wyss W 1991 *Found. Phys. Lett.* **4** 235